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Multivalent and Univalent Involutory Correspondences in a Plane determined by a Net of Curves of n^{th} Order.

A paper read before the New York Mathematical Society at the Meeting of February 6th, 1891.

By Chas. Proteus Steinmetz, *Yonkers, N. Y.*

Introduction.

[In the following, points will always be denoted by German letters, lines and curves by Latin letters; planes, pencils and nets by Greek letters.]

§1.—*Ranges of Points.*

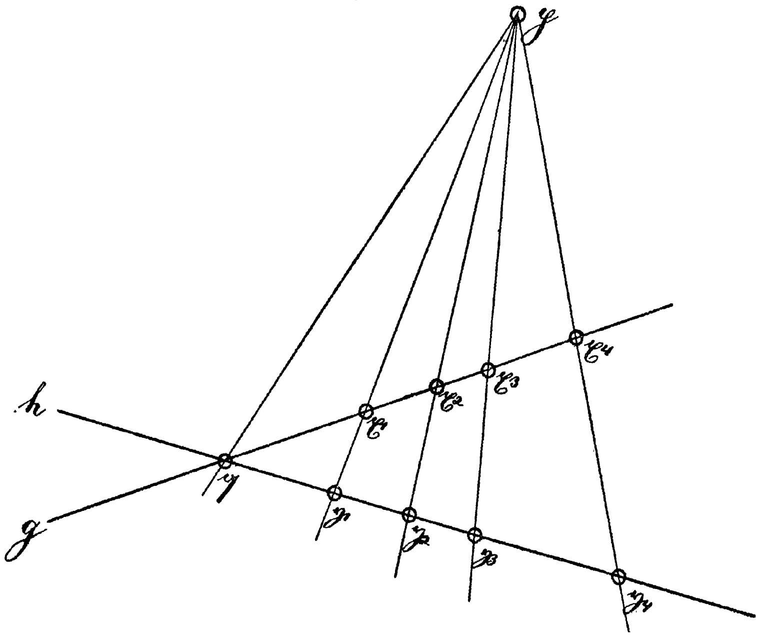

Fig. 1.

Let us assume two straight lines g and h. Then, between the points $\mathfrak{x}$ and

$\mathfrak{y}$ of these lines g and h, a correspondence can be established so that to any point $\mathfrak{x}$ on g corresponds a point $\mathfrak{y}$ on h.

The simplest way to produce such a correspondence is from a point $\mathfrak{p}$ taken at random, but outside of g and h, to draw a pencil of rays and attribute to each other the two points of intersection of any ray with the two lines g and h.

The correspondence established in this way is called "*perspective*," with the point $\mathfrak{p}$ as the "*centre of perspectivity*," and has the essential property that the *anharmonic ratio* of any four points $\mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_3\mathfrak{x}_4$ is the same as the anharmonic ratio of their corresponding points $\mathfrak{y}_1\mathfrak{y}_2\mathfrak{y}_3\mathfrak{y}_4$:

$$(\mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_3\mathfrak{x}_4) \barwedge (\mathfrak{y}_1\mathfrak{y}_2\mathfrak{y}_3\mathfrak{y}_4),$$

or

$$\frac{\mathfrak{x}_1\mathfrak{x}_3}{\mathfrak{x}_2\mathfrak{x}_3} \div \frac{\mathfrak{x}_1\mathfrak{x}_4}{\mathfrak{x}_2\mathfrak{x}_4} = \frac{\mathfrak{y}_1\mathfrak{y}_3}{\mathfrak{y}_2\mathfrak{y}_3} \div \frac{\mathfrak{y}_1\mathfrak{y}_4}{\mathfrak{y}_2\mathfrak{y}_4}.$$

At the point of intersection $\mathfrak{q}$ of the lines g and h lie two corresponding points, one in each range.

Analytically this correspondence is represented by a bilinear equation between the coordinates x and y of the points $\mathfrak{x}$ and $\mathfrak{y}$ of the ranges or forms g and h.

But if we reverse this reasoning and define the correspondence between the two ranges g and h by the condition that the anharmonic ratio of any four points $\mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_3\mathfrak{x}_4$ is the same as the anharmonic ratio of their corresponding points $\mathfrak{y}_1\mathfrak{y}_2\mathfrak{y}_3\mathfrak{y}_4$, then in general the connecting lines of corresponding points $\mathfrak{x}$ and $\mathfrak{y}$ do not intersect in a centre of perspectivity $\mathfrak{p}$, but envelop a conic curve, and the correspondence is called *projective* or *homographic*, so that perspectivity appears as a special case of projectivity.

But even two general projective ranges of points can be brought to perspectivity by merely moving them so that the point of intersection $\mathfrak{q}$ of the ranges g and h corresponds with itself.

Now, if these two ranges g and h coincide with the same line, then by the relation of projectivity a correspondence between the points of this line—considered as double line—is established so that to a point $\mathfrak{x}$, considered as a point of g, corresponds a point $\mathfrak{y}$ of h, and to this point $\mathfrak{y}$, considered as a point on g, corresponds again a point $\mathfrak{z}$ on h.

If, now, this point $\mathfrak{z}$ is again the same point $\mathfrak{x}$, so that to a point $\mathfrak{x}$, considered as point on g, corresponds on h the same point $\mathfrak{y}$, as to $\mathfrak{x}$, considered as point on h, corresponds on g, then this pair of points $\mathfrak{x}$, $\mathfrak{y}$ is said to be in *involution*, and it can be proved that when in two coincident projective ranges of points one pair of corresponding points is in involution, every pair of corresponding points $\mathfrak{x}$ and $\mathfrak{y}$ is in involution, and the correspondence between these coincident projective ranges of points is called an *involutory correspondence*, or in short an *involution.*

Then we see the individualities of those two coincident projective ranges of points g and h disappear, and the quadratic involution can be considered as such a combination or configuration of the points of one range, that to any point $\mathfrak{x}$ corresponds a point $\mathfrak{y}$, and to point $\mathfrak{y}$ again the point $\mathfrak{x}$.

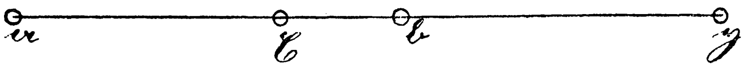

FIG. 2.

Such an involution is formed for instance by the third and fourth harmonic point. Because, when you assume 2 points $\mathfrak{a}$ and $\mathfrak{b}$, to a point $\mathfrak{x}$ always corresponds a point $\mathfrak{y}$ as fourth harmonic point with respect to $\mathfrak{a}$, $\mathfrak{b}$, and to $\mathfrak{y}$ corresponds as fourth harmonic point again the point $\mathfrak{x}$.

The points $\mathfrak{a}$ and $\mathfrak{b}$ are the *self-corresponding points* or *centre-points* of this involutory correspondence.

This harmonic range is a particular case of the general quadratic involution: the involution with real centre-points.

We call this involution a *univalent correspondence*, because to a point $\mathfrak{x}$ corresponds always one point $\mathfrak{y}$.

Analytically it is given by the roots of a quadratic equation, the coefficients of which are linear functions of a variable parameter.

A *multivalent involution* on a range of points will be such a correspondence that to one point $\mathfrak{x}$ corresponds a certain number of points $\mathfrak{y}_1, \mathfrak{y}_2 \dots \mathfrak{y}_{n-1}$, and to every point $\mathfrak{y}$ corresponds again the point $\mathfrak{x}$ and all the other points $\mathfrak{y}$, so that the points of the range are joined into groups of n, so that to a point $\mathfrak{x}$ of a group correspond the other $(n - 1)$ points of the same group, and inversely.

Such a correspondence is, for instance, produced by a linear pencil of curves of n^{th} order. Through any point $\mathfrak{x}$ of g passes one, and only one, curve of the pencil which intersects g in $(n - 1)$ additional points $\mathfrak{y}$ corresponding to $\mathfrak{x}$, and to any one of those points $\mathfrak{y}$ correspond again the other points $\mathfrak{y}$ together with $\mathfrak{x}$.

Therefore this correspondence is $(n-1)$-valent.

It contains $2(n-1)$ centre-points, or self-corresponding points, because $2(n-1)$ curves of the pencil of curves of n^{th} order touch the line g.

For further particulars on involutions in ranges of points see Weyr, Wiener Berichte, 1879.

§2.—*Planes in Correspondence.*

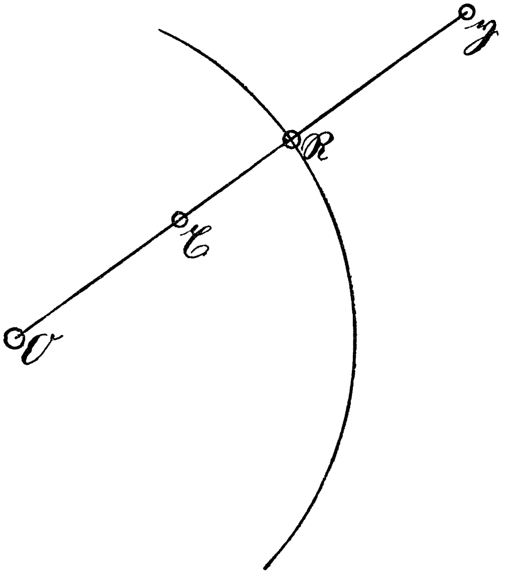

FIG. 3.

In the same way as between ranges of points, between two planes ε and η a correspondence of points, can be established so that to every point $\mathfrak{x}$ on ε corresponds one, or several points $\mathfrak{y}$ on η, and this correspondence can therefore be either *univalent* or *multivalent.*

If, now, the two planes ε and η coincide, to a point $\mathfrak{x}$, considered as point on ε, corresponds a point $\mathfrak{y}$ on η, and to $\mathfrak{y}$, considered as point on ε, corresponds again a point $\mathfrak{z}$ on η, and if this point $\mathfrak{z}$ is always the same as $\mathfrak{x}$, that is, if to a point $\mathfrak{x}$ as point on ε corresponds on η the same point $\mathfrak{y}$, as to $\mathfrak{x}$, considered as point on η, corresponds in ε, then the two coincident planes are said to be in *involutory coincidence* and produce an *involutory correspondence*; and again the individualities of the two different planes disappear.

Such an involutory correspondence of elementary character is, for instance, the correspondence by reciprocal radii.

Let us assume (Fig. 3) a point $\mathfrak{D}$ as centre. Then to every point $\mathfrak{x}$ is attributed on the ray $|\mathfrak{D}\mathfrak{x}|$ a point $\mathfrak{y}$ by the condition

$$|\mathfrak{D}\mathfrak{y}| = \frac{r^2}{|\mathfrak{D}\mathfrak{x}|},$$

or

$$|\mathfrak{D}\mathfrak{x}| \times |\mathfrak{D}\mathfrak{y}| = r^2,$$

where r^2 is a given quantity; and to $\mathfrak{y}$ corresponds again the point $\mathfrak{x}$.

All the points corresponding with themselves lie upon a circle with $\mathfrak{D}$ as centre and r as radius, which is called the *centre-curve* or *self-corresponding curve* of the correspondence; and this correspondence is called quadratic or of the 2^d order, because to the points of a straight line corresponds a circle, that is, a curve of 2^d order.

In this correspondence by reciprocal radii the points of the plane are joined into groups of two, and the connecting lines of corresponding points all going through the point $\mathfrak{D}$, the correspondence is *perspective*, with point $\mathfrak{D}$ as the *centre of perspectivity.*

Again, a *multivalent involutory correspondence* in the plane is a correspondence in which to a point $\mathfrak{x}$ belong a certain number of points $\mathfrak{y}_1, \mathfrak{y}_2 \dots \mathfrak{y}_{n-}$, and to any one of these points $\mathfrak{y}$ correspond the other points $\mathfrak{y}$, together with point $\mathfrak{x}$.

This multivalent involutory correspondence is $(n-1)$-valent, and joins the points of the plane into groups of n.

It is these multivalent and univalent involutory correspondences in the plane which I propose to consider here, the corresponding relations in spaces having been published in the "Zeitschrift für Mathematik und Physik," Dresden, 1890, p. 219.

CHAPTER I.

CORRESPONDENCES DEFINED BY A GENERAL NET.

§3.—*Definition and Quantivalence of the Correspondence.*

Let us assume in the plane a general net of curves of n^{th} order; that is, all those curves which, from three curves given at random—but which we first suppose to have no common point of intersection—are produced linearly, that is, by continued formation of pencils of curves.

Analytically, this net of curves is represented by the equation

$$F(x, y) \equiv \theta_1 f_1(x, y) + \theta_2 f_2(x, y) + \theta_3 f_3(x, y) = 0.$$

Then all the curves c of this net K, which intersect in one point $\mathfrak{x}$, form a pencil of curves of n^{th} order, and therefore intersect in $(n^2 - 1)$ additional points $\mathfrak{y}$, which, together with $\mathfrak{x}$, are the base-points of this pencil.

Therefore, to every point $\mathfrak{x}$ correspond $(n^2 - 1)$ points $\mathfrak{y}$.

On the other hand, to every one of those points $\mathfrak{y}$ correspond the point $\mathfrak{x}$ and the other $(n^2 - 2)$ points $\mathfrak{y}$.

Hence these points $\mathfrak{x} \ldots . \mathfrak{y}$ define an involutory

$$N = (n^2 - 1)\text{-valent}$$

correspondence of the points of the plane.

The locus of all those points $\mathfrak{x}$ which coincide with one of their corresponding points $\mathfrak{y}$, and thus correspond with themselves, is a curve of the $3(n-1)^{\text{th}}$ order $H^{3(n-1)}$, the *centre-curve* or *self-corresponding curve* of the correspondence. (Proof, see §6.)

§4.—*The Order of the Correspondence.*

To determine the order of the correspondence, we produce the curve C which corresponds to the straight line g.

For this purpose we make use of a representation of the curves c of n^{th} order of the net K by the straight lines l of the plane E, in attributing to each curve of n^{th} order c its linear or last polar-line with regard to a fixed point $\mathfrak{P}$ of the plane.*

Then to each line l of E corresponds a curve c of K, and to each curve c of K corresponds a straight line l of E.

To each pencil of curves c in K corresponds a pencil of rays in E, and inversely.

Hence to every point $\mathfrak{p}$ of K corresponds in E one point $\mathfrak{q}$, the base-point of the pencil of lines l in E, which corresponds to that pencil of curves c in K, which is determined by $\mathfrak{p}$ as one of its base-points.

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* But which point $\mathfrak{P}$ is supposed to be no particular point of the net K.

But, to every point $\mathfrak{q}$ in E correspond n^2 points $\mathfrak{p}$ in K, the base-points of the pencil of curves c corresponding to the pencil of lines with $\mathfrak{q}$ as centre.

This representation is therefore $1 \div n^2$-valent.

To a straight line g in K corresponds in E a curve k, which cuts any line l in the same number of points $\mathfrak{q}$, as g cuts the curve c, corresponding in K to the line l in E, that is, in n points.

Hence k is a (unicursal) curve of the n^{th} order.

To this curve $k^{(n)}$ corresponds in K a curve of the n^2th order, which consists of the line g and a curve $C^{(n^2-1)}$ of the $(n^2-1)^{\text{th}}$ order, the curve corresponding to g. That is—

"To every straight line g corresponds a curve $C^{(n^2-1)}$ of the order

$$M = (n^2 - 1),$$

as the locus of the groups of $(n^2 - 1)$ points $\mathfrak{y}$, corresponding to the points $\mathfrak{r}$ of the line g."

"The order of this involutory multivalent correspondence is: $M = (n^2 - 1)$; that is, the same as the quantivalence."*

§5.—*Points and Lines in the Correspondence.*

Each line g intersects the corresponding curve $C^{(n^2-1)}$ in $(n^2 - 1)$ points, of which

$3(n-1)$ points are centre-points or self-corresponding points, as points of intersection with the centre-curve $H^{3(n-1)}$.

The other $(n^2 - 1) - 3(n - 1) = (n - 1)(n + 2)$ points correspond to each other mutually:

"On every line there are $\frac{(n-1)(n+2)}{2}$ pairs of *conjugate* or *mutually corresponding points.*"

If a line g_0 contains more than $\frac{(n-1)(n+2)}{2}$ pairs of conjugate or mutually corresponding points, it corresponds to itself; that is, all the points of the line g_0

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* But these values of the order, and of the quantivalence of the correspondence, are derived only under the assumption of the non-existence of fundamental points, or common base-points, in the net, and do not hold if all the curves c of the net K contain common points of intersection. For this case see Chapter III.

correspond to one another, and the corresponding curve of g_0 is of the order $(n^2 - 2)$ only, or even of lower order.

"On two lines g_1 and g_2, given at random, there are $(n^2 - 1)$ pairs of corresponding points."

For g_2 intersects $C_1^{(n^2-1)}$ in $(n^2 - 1)$ points.

"On every line g there are $3(n - 1)(n^2 - 2)$ points which have a centre-point as a corresponding point."

For $C^{(n^2-1)}$ intersects $H^{3(n-1)}$ in $3(n - 1)(n^2 - 1)$ points, of which $3(n - 1)$ points lie on g.

Hence

"All the points $\mathfrak{x}$, which have a centre-point as corresponding point, produce a curve $K^{3(n-1)(n^2-2)}$ of the $3(n - 1)(n^2 - 2)$th order, which I call the *conjugate curve to the centre-curve* $H^{3(n-1)}$, because $K^{3(n-1)(n^2-2)}$ together with $H^{3(n-1)}$ corresponds to $H^{3(n-1)}$"

§6.—*The Centre-curve* $H^{3(n-1)}$ *of the Correspondence.*

Let $\mathfrak{z}$ be a centre-point, that is, a point which corresponds to itself, or rather to a point $\mathfrak{z}_1$ infinitely near to $\mathfrak{z}$. Then the line $t = |\mathfrak{z}\mathfrak{z}_1|$ is common tangent of all the curves c of the pencil determined by $\mathfrak{z}$.

Hence the centre-curve is the locus of contact-points of all the pencils with a common tangent, or of those points where all the curves of a pencil touch each other.

In such a pencil of tangent curves c, always one curve d is determined by a point $\mathfrak{z}_2$ infinitely near to the contact-point $\mathfrak{z}$, but outside of the common tangent $t = |\mathfrak{z}\mathfrak{z}_1|$.

This curve d has a node or double point in $\mathfrak{z}$.

Hence the centre-curve is the locus of the nodes of the curves c of the net K.

Now the first polar-curve of any point of the plane with respect to a given curve passes through all the nodes of this curve.

In a centre-point intersect therefore the first polar-curves of all the points of the plane, or what is the same, of three points not being in a straight line, with respect to that curve d of the net K, which has a node in this centre-point.

Now the first polar-curves of the points of the plane, taken with respect to the curves of nth order c of the net K, produce projective nets of curves of $(n - 1)$th order, which are projective also to the net K.

Hence the centre-curve H is the product of 3 projective nets of curves of $(n - 1)^{\text{th}}$ order, which are the first polar-curves of 3 points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, given at random, but so that they do not lie in a straight line, with respect to the curves of n^{th} order c of the net K.

Let these three projective nets be called A, B, Γ, their curves a, b, c.

Assume at random a straight line g. Every point $\mathfrak{p}$ of this straight line g determines a pencil α of curves a in net A, to which corresponds in B a pencil β of curves b. One of these curves b passes through $\mathfrak{p}$, and to this curve b corresponds in net Γ one curve c, which intersects g in $(n - 1)$ points $\mathfrak{q}$, so that to a point $\mathfrak{p}$ correspond $(n - 1)$ points $\mathfrak{q}$.

On the other hand, every point $\mathfrak{q}$ on g determines a pencil γ of curves c of $(n - 1)^{\text{th}}$ order of the net Γ, to which correspond two projective pencils α and β of curves a and b of the nets A and B.

These pencils of curves α and β, of $(n - 1)^{\text{th}}$ order, produce by their projective correspondence a curve of $2(n-1)^{\text{th}}$ order, which intersects g in $2(n-1)$ points $\mathfrak{p}$ corresponding to $\mathfrak{q}$. Thus:

to every point $\mathfrak{p}$ correspond $(n - 1)$ points $\mathfrak{q}$;
to every point $\mathfrak{q}$ correspond $2(n - 1)$ points $\mathfrak{p}$;

hence the coincident ranges of points $\mathfrak{p}$ and $\mathfrak{q}$ on line g have $3.(n - 1)$ double-points or self-corresponding points,* which, as points of intersection of 3 corresponding first polar-curves a, b, c, are points of the centre-curve H. That is,

"The centre-curve $H^{3(n-1)}$ of the correspondence is of the $3(n - 1)^{\text{th}}$ order."

CHAPTER II.

CORRESPONDENCES DEFINED BY A NET WITH FIXED BASE-POINTS OR FUNDAMENTAL POINTS.

§7.—*Definition and Quantivalence.*

If all the curves c of n^{th} order of the net K pass through the same point $\mathfrak{P}$, there intersecting each other, we call this point $\mathfrak{P}$ a (*simple*) *fundamental point* of the correspondence.

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*In general, μ-projective $(\mu - 1)$-dimensional linear systems of curves of m^{th} order produce as the locus of their common points of intersection of corresponding curves a curve of the order μm, and therefore, for $m = n - 1$, $\mu = 3$, $3(n - 1)$.

If all the curves c pass through the same point $\mathfrak{Q}$, and have this point as a $\varkappa$-ple point, we call this point $\mathfrak{Q}$ a *fundamental point of the correspondence of multiplicity* $\varkappa$, or a $\varkappa$*-ple fundamental point.*

At a simple fundamental point $\mathfrak{P}$, one curve c has a node or double-point, but in general, at a $\varkappa$-ple fundamental point no curve c has a multiple-point of order higher than $\varkappa$.

Therefore all fundamental points are points of the centre-curve H.

Every $\varkappa$-ple fundamental point is a $\varkappa^2$-ple point of intersection of the curves c of the net K, and therefore consumes $\varkappa^2$ points $\mathfrak{y}$, corresponding to every point $\mathfrak{x}$ of the plane. Hence

"The quantivalence of the correspondence is

$$N = n^2 - \Sigma \varkappa_i^2 - 1,$$

where $\varkappa_i$ is the multiplicity of the fundamental point $\mathfrak{Q}_i$."

§8.—*Singular Curves and Singular Points.*

"Of fundamental points of a higher multiplicity than $n/2$, only one can exist. Then all the other fundamental points must be of a multiplicity equal to or less than $n - \varkappa$, where $\varkappa$ is the multiplicity of the first point."

"If the fundamental points $\mathfrak{Q}_1, \mathfrak{Q}_2 \dots . \mathfrak{Q}_i$ of the multiplicities $\varkappa_1, \varkappa_2 \dots . \varkappa_i$ lie upon a curve $S^{(m)}$ of the m^{th} order, and have for this curve $S^{(m)}$ the multiplicities $\theta_1, \theta_2 \dots . \theta_i$, we must have

$$\Sigma \varkappa_i \theta_i \leqq mn."$$

Otherwise this curve $S^{(m)}$ would be a part of all the curves c of the net K, and this net K therefore would reduce to a net of $(n - m)^{\text{th}}$ order.

A special case is—

"The sum of the multiplicities of all the fundamental points lying on a straight line must be equal to or less than n."*

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*In the corresponding relations in space these conditions do not exist, but fundamental curves and fundamental lines, that is, curves and lines common to all the surfaces of n^{th} order of the tri-dimensional system which defines the correspondence, can exist, and indeed must exist, if the correspondence is univalent.

"If the fundamental points $\mathfrak{O}_i$ of the multiplicities $\varkappa_i$ lie on a curve $S^{(m)}$ of the m^{th} order, and are for this curve of the multiplicity θ_i, and if

$$\Sigma \varkappa_i \theta_i = mn,$$

we call this curve $S^{(m)}$ a *singular curve of* m^{th} *order of the correspondence.*"

The pencil of curves c, determined by a point of a singular curve of m^{th} order $S^{(m)}$, consists of this curve $S^{(m)}$ and of a pencil of curves of $(n-m)^{\text{th}}$ order which intersect in

$$(n-m)^2 - \Sigma \varkappa_i'^2 - \Sigma (\varkappa_i'' - \theta_i)^2$$

points (where $\varkappa_i'$ is the multiplicity of those fundamental points $\mathfrak{O}_i'$ which do not lie on $S^{(m)}$, and $\varkappa_i''$ the multiplicity of those fundamental points $\mathfrak{O}_i''$ which are θ_i-ple points for $S^{(m)}$). These points we term the *singular points* $\mathfrak{S}$, *associated with the singular curve* $S^{(m)}$.

To any point of a singular curve $S^{(m)}$ corresponds the whole singular curve $S^{(m)}$ and their associated singular points $\mathfrak{S}$.

To a singular point $\mathfrak{S}$ correspond its other associated singular points and the whole associated singular curve $S^{(m)}$.

Therefore every singular curve $S^{(m)}$ is a part of the centre-curve H of the correspondence, but not in general any singular point.

Hence the order of the centre-curve H is

$$3(n-1) - \Sigma m_i,$$

and

"The sum of the orders of the singular curves $S^{(m)}$ is equal to or less than $3(n-1)$,

$$\Sigma m_i \leqq 3(n-1)."$$

§9.—*The Order of the Correspondence.*

The fundamental points have no influence upon the order of the correspondence, and, if no singular curves exist, the order of the correspondence is

$$M = n^2 - 1.$$

Every line g intersects a singular curve $S^{(m)}$ of m^{th} order in m points, to every one of which the whole curve $S^{(m)}$ corresponds, together with its singular points.

Hence, if a singular curve of m^{th} order exists, the curve corresponding to a straight line g consists of the curve $S^{(m)}$, counted m-fold, and a curve C of the $(n^2 - m^2 - 1)^{\text{th}}$ order. Thus:

"The order of the correspondence is

$$M = n^2 - \Sigma m_i^2 - 1, .$$

where m_i is the order of the singular curves $S^{(m)}$."

§10.—*The General Curve C.*

"To a general straight line g corresponds a curve C of the

$$M = (n^2 - \Sigma m_i^2 - 1)^{\text{th}} \text{ order},$$

which has every $\varkappa_i$-ple fundamental point $\mathfrak{Q}_i$ as a

$$\nu = n\varkappa_i\text{-ple point},$$

if $\mathfrak{Q}_i$ does not lie on a singular curve, but only as a

$$\nu' = (n\varkappa_i' - \Sigma m_\sigma \theta_\sigma)\text{-ple point},$$

if $\mathfrak{Q}_i'$ lies on the singular curves $S_\sigma^{(m)}$ and is a θ_σ-ple point for those curves."

This curve C has every singular point $\mathfrak{S}$, which is associated with a singular curve $S^{(m)}$ of m^{th} order, as a

$$\nu'' = m\text{-ple point},$$

if $\mathfrak{S}$ does not lie on any singular curve, but only as a

$$\nu''' = (m - \Sigma m_\sigma \theta_\sigma)\text{-ple point},$$

if $\mathfrak{S}$ is a θ_σ-ple point of the singular curve $S^{(m)}$.

Between these numbers ν, ν', ν'', ν''', the quantivalence N and the order M exists the relation

$$\Sigma (\nu^2 + \nu'^2 + \nu''^2 + \nu'''^2) = (M + 1)(M - N).$$

To prove these results, we make use of a construction for the curve C different from that given in §4.

We produce the curve C, corresponding to a straight line g, by two pencils α and β of curves a and b of the net K, which pencils a and b, as belonging to the same net, have one curve c in common.

These curves a and b have with each other an n-valent correspondence by means of the points of the line g; to every curve a corresponding to the n-curves b, which are determined by the points of intersection of a with g.

Now these n-curves b intersect a in n^3 points. n of these are points of the line g, and the other $n^3 - n = n(n^2 - 1)$ points are the points of intersection of the curve a with the curve C, and, a being of the n^{th} order, C therefore is of the $(n^2 - 1)^{\text{th}}$ order.

If $S^{(m)}$ is a singular curve of m^{th} order, it intersects a in nm points. Each one of these is a m-ple point of the locus corresponding to g, because corresponding to m points of g, and the real curve C has therefore only $n(n^2 - m - 1)$ points of intersection with a; that is, C is of the $(n^2 - m - 1)^{\text{th}}$ order.

Now, let $\mathfrak{Q}_i$ be a fundamental point of multiplicity $\varkappa_i$, then it is $\varkappa$-ple point of intersection of a curve a with any curve b. Therefore it is a $n\varkappa_i^2$-ple point of intersection of a with the n corresponding curves b, that is, with the curve C, and, $\mathfrak{Q}_i$ being $\varkappa_i$-ple point of a, it is $n\varkappa_i$-ple point of C. But if it is a θ-ple point of a singular curve $S^{(m)}$ of m^{th} order, it is $m\theta\varkappa_i$-ple point of intersection of a with the curve $S^{(m)}$, counted m-fold, and therefore only a $(n\varkappa_i^2 - m\theta\varkappa_i)$-ple point of intersection of a with C, that is, a $(n\varkappa_i - m\theta)$-ple point of C, etc.

§11.—*Particular Curves C.*

If a line g passes through a ρ-ple fundamental point $\mathfrak{R}$, the pencils a and b are put in a correspondence only $(n - \rho)$-valent, because every curve a intersects g in only $(n - \rho)$ variable points. If g passes through a singular point $\mathfrak{W}$ associated with a singular curve of w^{th} order, $S^{(w)}$, this curve $S^{(w)}$ disintegrates from the curve C corresponding to g, and in considering the number of points of intersection coinciding with a fundamental point, or with a singular point, we get in the same way as in §10:

"To a straight line g, passing through the fundamental points $\mathfrak{R}_\mu$ of $\rho_\mu{}^{\text{th}}$ multiplicity and the singular points $\mathfrak{W}_\tau$, which are associated with the singular curves $S_\tau^{(w)}$ of $w_\tau{}^{\text{th}}$ order, corresponds a curve C of the

$$M = \big(n(n - \Sigma\rho_\mu) - \Sigma m_i(m_i - \theta_i) - \Sigma w_\tau - 1\big)^{\text{th}} \text{ order,}$$

where θ_i is the multiplicity of $\mathfrak{R}_\mu$ for S_i. And this curve, at a fundamental point $\mathfrak{Q}_i$ of the multiplicity $\varkappa_i$, which fundamental point is θ_σ-ple point for the singular curves $S_\sigma^{(m)}$, and θ_τ-ple point for the singular curves $S_\tau^{(w)}$, has a

$$\nu = \big((n - \Sigma\rho_\mu)\varkappa_i - \Sigma_\sigma m\theta_\sigma - \Sigma\theta_\tau\big)\text{-ple point};$$

at a point $\mathfrak{R}_\mu$ it has a

$$\nu' = \big((n - \Sigma\rho_\mu)\rho_\mu - \Sigma m_\sigma\theta_\sigma - \Sigma\theta_\tau - 1\big)\text{-ple point};$$

at a singular point $\mathfrak{S}_i$ it has a

$$\nu'' = (m_i - \Sigma m_\sigma \theta_\sigma)\text{-ple point,}$$

where m_i is the order of the singular curve $S_i^{(m)}$, associated with the singular point $\mathfrak{S}_i$, which is a θ_σ-ple point for the singular curve $S_\sigma^{(m)}$; finally, at a singular point $\mathfrak{S}_{\mathfrak{w}}$, associated with a point $\mathfrak{W}_\tau$, C has a

$$\nu''' = (w_\tau - \Sigma m_\sigma \theta_\sigma + 1)\text{-ple point.}$$

Between these numbers ν, ν', ν'', ν''' and ν_1, ν_1', ν_1'', ν_1''' of two curves C and C_1 of orders M and M_1 respectively, and the quantivalence N, exists the relation

$$\Sigma(\nu\nu_1 + \nu'\nu_1' + \nu''\nu_1'' + \nu'''\nu_1''') + N$$
$$= MM_1 - (M - s)(N - 1) = MM_1 - (M_1 - s_1)(N - 1),$$

where s is the sum of the multiplicities of the fundamental and singular points on g_1 for C."

As a particular case of this theorem, we have—

"To a straight line s, which connects the fundamental points $\mathfrak{Q}_i$ of multiplicity $\varkappa_i$ so that $\Sigma\varkappa_i = n$, corresponds a curve of O^{th} order, which consists merely of a number of separate, singular points associated to the line s as a singular line."

§12.—*The Centre-curve H.*

"The centre-curve H is of the order

$$3(n - 1) - \Sigma m_i,$$

and contains any $\varkappa$-ple fundamental point $\mathfrak{Q}$ which is of the multiplicity θ_σ for the singular curves $S_\sigma^{(m)}$, if $\varkappa > 1$, as a

$$(3\varkappa - \Sigma\theta_\sigma - 2)\text{-ple point,}$$

but a simple fundamental point as a double-point or node, if no singular curve passes through it, otherwise as a point of the multiplicity

$$2 - \Sigma\theta_\sigma."$$

The latter, because the simple fundamental point is a double-point for one curve of the net, while the $\varkappa$-ple fundamental point in general is not a $(\varkappa + 1)$-ple point of any curve c of the net K.

"In a simple fundamental point, the centre-curve H has the same two tangents as the one curve of the net which has this point as double-point."

We prove these results by producing the centre-curve H in the same way as in §6, by three projective nets of first polar-curves, but by using as the point a the fundamental point $\mathfrak{Q}$ and drawing line g through $\mathfrak{Q}$.

"The sum of the orders of the singular curves $S_i^{(m)}$ is

$$\Sigma m_i \leqq 3\,(n-1)."$$

"If

$$\Sigma m_i = 3\,(n-1),$$

the correspondence has no centre-curve or self-corresponding curve, but only *separate centre-points* or *self-corresponding points*."

N. B.—In the foregoing we did not specially mention the case that all the curves of the net K had a certain number of *common tangents in a fundamental point*, because this case affords nothing new. For any fundamental point with one or more common tangents can be considered as consisting of, and dissolved into, two or more separate fundamental points lying infinitely near each other.

CHAPTER III.

RESEARCH ON THE EXISTENCE OF UNIVALENT CORRESPONDENCES.

§13.—*Fixed Base-points and Elements of Determination.*

A net of curves of n^{th} order is determined by

$$P = \frac{(n+1)(n+2)}{2} - 3 = \frac{(n-1)(n+4)}{2} \text{ points or elements,}$$

and can contain in maximo

$$Q = n^2 - 2 \text{ fixed points of intersection.}$$

Now one simple fundamental point represents 1 element of determination and 1 point of intersection.

One double fundamental point represents 3 elements of determination and 4 points of intersection.

One triple fundamental point represents 6 elements of determination and 9 points of intersection.

One $\varkappa$-ple fundamental point represents $\frac{\varkappa(\varkappa+1)}{2}$ elements of determination and $\varkappa^2$ points of intersection.

The correspondence is univalent when the number of fixed points of intersection is

$$\Sigma\varkappa_i^2 = Q = n^2 - 2.$$

Then must be either

$$\Sigma\frac{\varkappa_i(\varkappa_i+1)}{2} = P = \frac{(n-1)(n+4)}{2},$$

and the correspondence is fully determined by its fundamental points, or

$$\Sigma\frac{\varkappa_i(\varkappa_i+1)}{2} < P = \frac{(n-1)(n+4)}{2},$$

and the correspondence is not fully determined by its fundamental points.

§14.—*Univalent Correspondences produced by Simple Fundamental Points.*

That a univalent correspondence might be produced by simple fundamental points alone, the condition is

$$\frac{(n-1)(n+4)}{2} \leqq n^2 - 2,$$

hence

$$n \leqq 3:$$

"By simple fundamental points univalent correspondences can be produced only by means of nets of conics and cubics."

$n = 1$: no correspondence defined.

$n = 2$: 2 simple fundamental points. $P = 3$. $\Sigma\frac{\varkappa_i(\varkappa_i+1)}{2} = 2$. The correspondence is not yet defined by the fundamental points alone.

$n = 3$: 7 simple fundamental points. $P = 7$. $\Sigma\frac{\varkappa_i(\varkappa_i+1)}{2} = 7$. The correspondence is just determined by its fundamental points.

§15.—*Univalent Correspondences produced by Multiple Fundamental Points.*

If a fundamental point is of higher multiplicity than $n/2$, only one of this kind of fundamental points can exist, and the other fundamental points must be of a multiplicity equal or lower than $n - \varkappa$.

"It is always possible to determine univalent correspondences by means of nets of curves of n^{th} order with $\varkappa$-ple fundamental points, if $\varkappa > n/3$."

For let the correspondence contain δ $\varkappa$-ple points, where $\varkappa < n/2$. Then these δ $\varkappa$-ple points represent

$$\delta \frac{\varkappa(\varkappa+1)}{2} \text{ elements of determination}$$

and

$$\delta\varkappa^2 \text{ fixed points of intersection.}$$

Now, in neglecting at first the condition that δ is an integer number, and supposing all fundamental points to be $\varkappa$-ple points, the condition of univalence is

$$\delta\varkappa^2 = n^2 - 2,$$

but

$$\delta \frac{\varkappa(\varkappa+1)}{2} \leqq \frac{(n-1)(n+4)}{2},$$

whence

$$\varkappa \geqq \frac{n^2-2}{3n-2}.$$

That is,

"The multiplicity of the fundamental points must be equal to or higher than $\frac{n^2-2}{3n-2}$ to determine a univalent correspondence."

Now, in excluding the case

$$n \leqq 3$$

as considered already, we get

$$\frac{n^2-2}{3n-2} < n/3.$$

Therefore

"$\varkappa > n/3$ determines univalent correspondences."*

N. B.—Very different conditions exist in space. There, by a tri-dimensional system of surfaces of n^{th} order, it is impossible to produce univalent involutory correspondences by mere simple or multiple fundamental points (except when $n = 2$), but only the existence of fundamental curves, that is, curves which are common curves of intersection of all the surfaces of n^{th} order of the tri-dimensional system, produces univalent correspondences in space.

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* This proof is not rigorously correct because we neglected the condition that δ is an integer number. But, by a slight modification, this condition can be accounted for.

CHAPTER IV.

PARTICULAR CORRESPONDENCES.

A.—UNIVALENT CORRESPONDENCES.

§16.—*Perspective Quadratic Correspondence*, $n=2$.

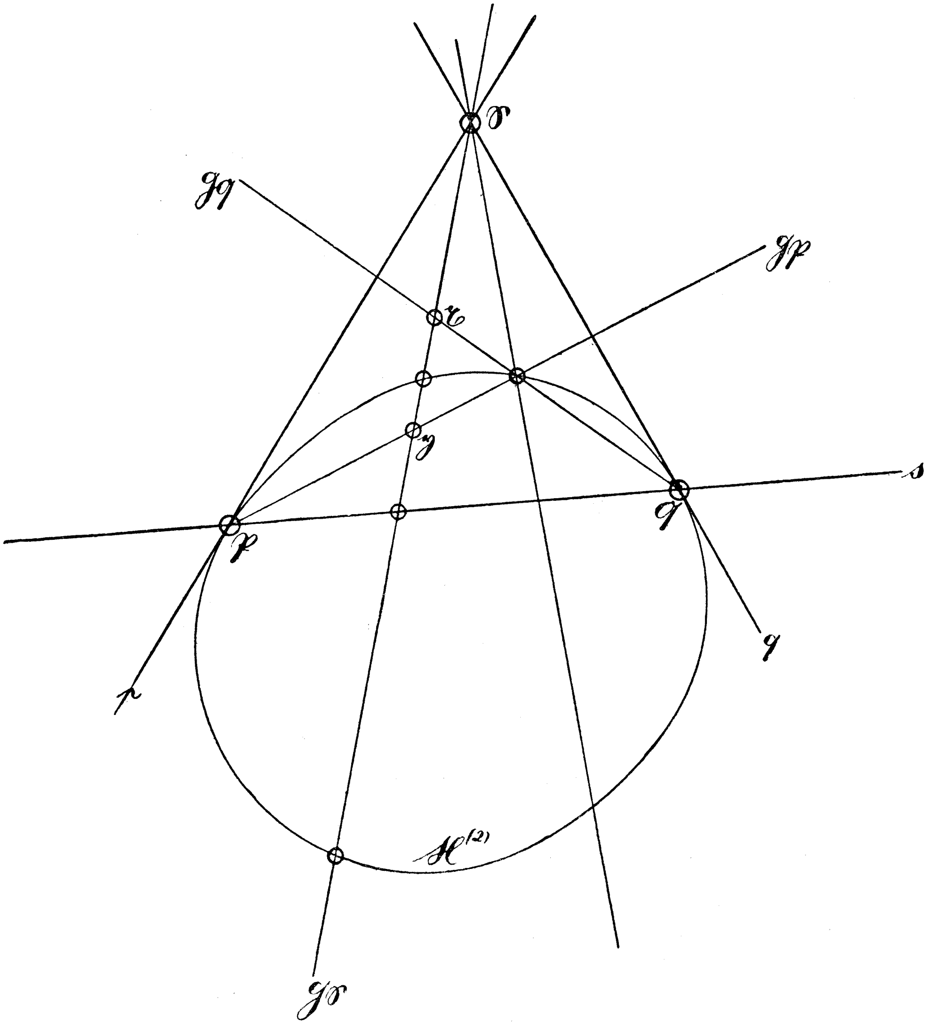

FIG. 4.

A net of conics, which have two points $\mathfrak{P}$ and $\mathfrak{Q}$ as common points of intersection, determines a univalent correspondence, because any point $\mathfrak{x}$ determines a conic pencil which has an additional base-point $\mathfrak{y}$.

This correspondence is not fully determined by its fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$.

The line joining the fundamental points, $s = |\mathfrak{P}\mathfrak{Q}|$, is *singular line* of the correspondence, and therefore corresponds to every line of the plane.

Hence the correspondence is *quadratic.*

With the singular line s is associated as corresponding point the centre $\mathfrak{S}$ of a pencil of rays, which, together with s, form a particular conic pencil of the net. (Fig. 4.)

The *centre-curve* of the net being of the third order, consists of the singular line s and a *conic* $H^{(2)}$, which passes through the fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$, and there has as tangents the lines $p = |\mathfrak{S}\mathfrak{P}|$ and $q = |\mathfrak{S}\mathfrak{Q}|$. Hence the singular line s is the polar-line to the singular point $\mathfrak{S}$ with regard to the centre-conic $H^{(2)}$.

To a straight line g corresponds a conic, which passes through the fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$ and through the singular point $\mathfrak{S}$.

To the rays $g_{\mathfrak{P}}$ of the pencil with its centre at the fundamental point $\mathfrak{P}$ correspond the rays $g_{\mathfrak{Q}}$ of the pencil with its centre at the other fundamental point $\mathfrak{Q}$, and inversely. These two corresponding pencils of rays $\mathfrak{P}$ and $\mathfrak{Q}$ are *projective,* and the locus of their points of intersection is the centre-conic $H^{(2)}$.

The rays $g_{\mathfrak{S}}$ of the pencil, with the singular point $\mathfrak{S}$ as the centre, are cut by the corresponding pencils $\mathfrak{P}$ and $\mathfrak{Q}$ in corresponding points $\mathfrak{x}$ and $\mathfrak{y}$, which have the points of intersection of $g_{\mathfrak{S}}$ with $H^{(2)}$ as self-corresponding points, and $\mathfrak{S}$ and $(s, g_{\mathfrak{S}})$ as a pair of corresponding points, and therefore lie in *quadratic involution.*

Thus this quadratic correspondence is *perspective,* with the *singular point* $\mathfrak{S}$ as the *centre of perspectivity.*

The rays of the pencils $\mathfrak{P}$ and $\mathfrak{Q}$ correspond to one another, those of the pencil $\mathfrak{S}$ correspond to themselves and contain quadratic involutions of corresponding points.

To the line s corresponds point $\mathfrak{S}$, and conversely.
" " p " " $\mathfrak{P}$, " "
" " q " " $\mathfrak{Q}$, " "

The corresponding curve of a conic is a unicursal quartic, which has the points $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{S}$ as its double-points.

The corresponding curve of a conic, which passes through one fundamental point $\mathfrak{P}$, is a unicursal cubic, which passes through the points $\mathfrak{P}$ and $\mathfrak{S}$, and has the other fundamental point $\mathfrak{Q}$ as its double-point.

The corresponding curve of a conic, which passes through the singular point $\mathfrak{S}$, is a unicursal cubic, which passes through the fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$, and has the singular point $\mathfrak{S}$ as its double point.

The corresponding curve of a conic, which passes through both fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$, is another conic which touches the former one in the fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$.

The corresponding curve of a conic, which passes through the singular point $\mathfrak{S}$ and one of the fundamental points $\mathfrak{P}$, is another conic which passes through the other fundamental point $\mathfrak{Q}$ and through the singular point $\mathfrak{S}$.

The corresponding curve of a conic, which passes through the three points $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{S}$, is a straight line, and conversely.

This correspondence is determined by the centre-conic $H^{(2)}$ and the singular point $\mathfrak{S}$. The singular line s is the polar-line of $\mathfrak{S}$ with respect to the centre-conic $H^{(2)}$, and any pair of corresponding points forms an harmonic range with the self-corresponding points $(g_{\mathfrak{S}}, H^{(2)})$, on the rays $g_{\mathfrak{S}}$ of the pencil $\mathfrak{S}$.

§17.—*Correspondence by Reciprocal Radii.*

Of special interest is a particular case of this perspective quadratic correspondence which we derive by assuming as fundamental point $\mathfrak{P}^{(i)}$ and $\mathfrak{Q}^{(i)}$ the two *imaginary circular-points at infinity.*

Then all the conics of the net are *circles.*

The *singular line* is the *straight line* s_∞ *at infinity.*

The *centre-conic* $H^{(2)}$ is a *circle*, because passing through the circular-points $\mathfrak{P}^{(i)}$ and $\mathfrak{Q}^{(i)}$ as fundamental points, which has the singular point $\mathfrak{S}$ *as its centre,* because $\mathfrak{S}$ is the pole of the singular line s_∞ at infinity, with respect to the centre-circle $H^{(2)}$.

Let the radius of this centre-circle $= r$.

Any pair of corresponding points $\mathfrak{x}$, $\mathfrak{y}$ forms an harmonic range on a ray $g_{\mathfrak{S}}$, which has the points of intersection of $|\mathfrak{x}\mathfrak{y}| = g_{\mathfrak{S}}$ with the centre-circle $H^{(2)}$ as double-points.

Hence, from the properties of the harmonic range, we derive

$$\mathfrak{S}\mathfrak{x} \times \mathfrak{S}\mathfrak{y} = r^2,$$

the *definition of the correspondence by reciprocal radii.*

To a straight line g corresponds a conic through $\mathfrak{S}$, $\mathfrak{P}^{(i)}$, $\mathfrak{Q}^{(i)}$, that is, a circle through $\mathfrak{S}$ which intersects g upon $H^{(2)}$.

To a circle, that is, a conic through $\mathfrak{P}^{(i)}$ and $\mathfrak{Q}^{(i)}$, corresponds another conic through $\mathfrak{P}^{(i)}$, $\mathfrak{Q}^{(i)}$; that is, a circle.

To a circle through $\mathfrak{S}$ corresponds a straight line, etc.

In this way we derive the properties of the well-known circular correspondence by particularization of the general perspective quadratic correspondence into which it can be transformed by collinear transformation.

§18.—*Correspondence of* 8^{th} *Order*, $n = 3$.

A net of cubics, with 7 common base-points, $\mathfrak{A}$, $\mathfrak{B}$ .... $\mathfrak{G}$, as fundamental points, defines a univalent correspondence which is just determined by its fundamental points.

This correspondence is of the 8th order, and to any straight line corresponds a unicursal curve of 8th order, which has the 7 fundamental points $\mathfrak{A}$, $\mathfrak{B}$.... $\mathfrak{G}$ as triple-points.

The corresponding curve of a straight line, which passes through one of the fundamental points, $\mathfrak{A}$, is a unicursal quintic which passes through $\mathfrak{A}$ as simple point, and has the other 6 fundamental points $\mathfrak{B}$ .... $\mathfrak{G}$ as double-points.

The corresponding curve of one of the 21 connecting lines of two fundamental points, is the conic determined by the other 5 fundamental points.

The centre-curve $H^{(6)}$ of the correspondence is of the 6th order, with the 7 fundamental points $\mathfrak{A}$, $\mathfrak{B}$ .... $\mathfrak{G}$ as double-points.

Here again a particular case of the correspondence is of special interest.

§19.—*The Quadratic Steinerian Correspondence.*

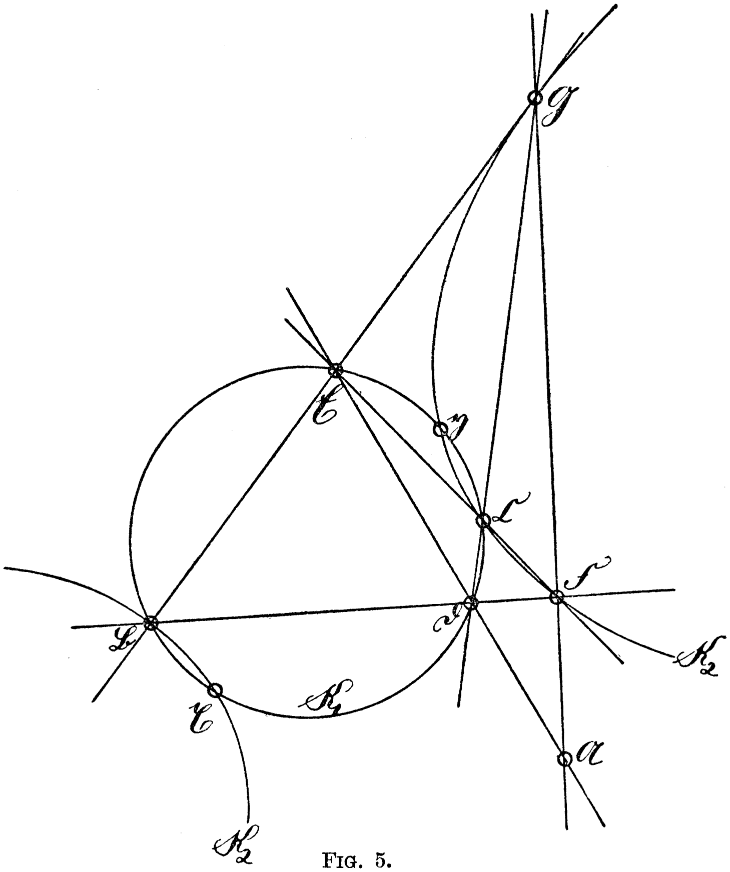

FIG. 5.

Let the 7 fundamental points $\mathfrak{A}$, $\mathfrak{B}$ . . . . $\mathfrak{G}$ of the correspondence considered in §18 lie three upon each of 6 straight lines, so that the configuration of these 7 points forms a *complete quadrangle.*

Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be the *diagonal points,* $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ the vertices of this quadrangle. Then those 6 lines

$$|\mathfrak{A}\mathfrak{D}\mathfrak{E}|, \quad |\mathfrak{A}\mathfrak{F}\mathfrak{G}|,$$
$$|\mathfrak{B}\mathfrak{D}\mathfrak{F}|, \quad |\mathfrak{B}\mathfrak{G}\mathfrak{E}|,$$
$$|\mathfrak{C}\mathfrak{D}\mathfrak{G}|, \quad |\mathfrak{C}\mathfrak{E}\mathfrak{F}|,$$

disintegrate from all the curves of 8th order of the correspondence as *singular lines.*

Hence the corresponding curve of a straight line is a conic which passes through the diagonal points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, for these points are triple-points in the general $C^{(8)}$, but only double-points in the quadrangle. That is,

"The correspondence is *quadratic.*"

From the centre-curve $H^{(6)}$ these 6 singular lines disintegrate also, and the general centre-curve being of 6th order, we derive

"This quadratic correspondence has *no centre-curve*, but only 4 *separate self-corresponding points*, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$."

For these 4 points are triple-points in the quadrangle, but only double-points in the general centre-curve.

Every pencil of cubics of this particular net contains 6 special cubics which consist of a singular line and a conic passing through the 4 fundamental points not lying on this singular line.

Any two of those conics have two fundamental points in common.

Hence, to construe the point $\mathfrak{y}$, which corresponds to $\mathfrak{x}$ taken at random in the plane, we make use of these particular cubics of the cubic pencil determined by $\mathfrak{x}$, and thus get the point $\mathfrak{y}$ as fourth point of intersection of two conics, which have the point $\mathfrak{x}$ and two fundamental points in common.

Therefore the construction of $\mathfrak{y}$ is linear.

Thus we get the point $\mathfrak{y}$, which corresponds to $\mathfrak{x}$, as the fourth point of intersection of two conics, which are parts of cubics of the cubic pencil $\mathfrak{x}$, the other part of these special cubics being singular lines.

Let these two special cubics of the cubic pencil $\mathfrak{x}$ be

1). The singular line $|\mathfrak{A}\mathfrak{F}\mathfrak{G}|$ and the conic K_1 passing through $\mathfrak{x}\mathfrak{B}\mathfrak{C}\mathfrak{D}\mathfrak{E}$.
2). " " " $|\mathfrak{A}\mathfrak{D}\mathfrak{E}|$ " " K_2 " " $\mathfrak{x}\mathfrak{B}\mathfrak{C}\mathfrak{F}\mathfrak{G}$.

These two conics K_1 and K_2 intersect in $\mathfrak{x}\mathfrak{B}\mathfrak{C}$ and the corresponding point $\mathfrak{y}$.

Now, by the projective properties of the conic, we get in the conic K_1 the projective pencils of rays, with the centres $\mathfrak{B}$ and $\mathfrak{C}$,

$$\mathfrak{B}(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}) \barwedge \mathfrak{C}(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}), \tag{a}$$

in the conic K_2 the projective pencils of rays, with the same centres $\mathfrak{B}$ and $\mathfrak{C}$,

$$\mathfrak{B}(\mathfrak{x}\mathfrak{y}\mathfrak{F}\mathfrak{G}) \barwedge \mathfrak{C}(\mathfrak{x}\mathfrak{y}\mathfrak{F}\mathfrak{G}), \tag{b}$$

or, what is the same,

$$\mathfrak{B}(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}) \barwedge \mathfrak{C}(\mathfrak{x}\mathfrak{y}\mathfrak{E}\mathfrak{D}); \tag{c}$$

hence (c) combined with (a),

$$\mathfrak{C}(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}) \barwedge \mathfrak{C}(\mathfrak{x}\mathfrak{y}\mathfrak{E}\mathfrak{D}),$$

therefore

$$\mathfrak{C}(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}) = -1,$$

or $\mathfrak{C}(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E})$ forms an harmonic range; that is,

"The pairs of corresponding points $\mathfrak{x}$, $\mathfrak{y}$ of this quadratic involutory correspondence are projected from any diagonal point $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ by the rays of a quadratic involution, which has as its self-conjugates the two singular lines intersecting in the diagonal point."

Now the construction of the corresponding point $\mathfrak{y}$ is the following: Connect $\mathfrak{x}$ with two diagonal points $\mathfrak{B}$ and $\mathfrak{C}$ and produce the fourth harmonic rays with respect to the pairs of singular lines intersecting in $\mathfrak{B}$ and $\mathfrak{C}$ to $|\mathfrak{B}\mathfrak{x}|$ and $|\mathfrak{C}\mathfrak{x}|$. These fourth harmonic rays intersect in $\mathfrak{y}$, which corresponds to $\mathfrak{x}$ in this quadratic correspondence.

But this is the construction of the *Steinerian quadratic involutory correspondence*, which has the diagonal fundamental points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ as its base-points, and the four other fundamental points $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ as its self-corresponding points, or as the base-points of that conic pencil, the conjugate poles of which are the corresponding points of the Steinerian correspondence, so that we need not enter further into the consideration of this particular correspondence.

Therefore, we get a new property of the Steinerian correspondence—

"Any pair of corresponding points of a Steinerian correspondence gives with the three base-points and the four self-corresponding points of the correspondence, a set of 9 base-points of a cubic pencil."

§20.—*Correspondence of 5th Order*, $n = 3$.

Another univalent correspondence is defined by a net of cubics with one double fundamental point $\mathfrak{D}$ and three simple fundamental points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, but is not fully determined by its fundamental points.

This correspondence is of the 5th order, and contains three singular lines, $|\mathfrak{D}\mathfrak{A}|$, $|\mathfrak{D}\mathfrak{B}|$, $|\mathfrak{D}\mathfrak{C}|$, every one of which is associated with one singular point, $\mathfrak{A}_1$, $\mathfrak{B}_1$, $\mathfrak{C}_1$.

The centre-curve is a cubic $H^{(3)}$, passing through $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ as simple points.

The corresponding curve of a straight line is a unicursal quintic, which has the point $\mathfrak{D}$ as triple-point, the points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ as double-points, and passes through the singular points $\mathfrak{A}_1$, $\mathfrak{B}_1$, $\mathfrak{C}_1$ as simple points.

§21.—*Nets of Quartics.*

Number of fixed points of intersection, $Q = n^2 - 2 = 14$.

Number of elements of determination, $P = \dfrac{(n-1)(n+4)}{2} = 12$.

The quartic net gives the following univalent correspondences:

1). 2 double-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, and 6 simple fundamental points, $\mathfrak{P}_i$.

The correspondence is fully determined thereby.

14th Order.—Centre-curve, $H^{(8)}$ of 8th order, with 2 triple-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, and 6 double-points, $\mathfrak{P}_i$.

One singular line, $s = |\mathfrak{D}_1\mathfrak{D}_2|$, with one associated singular point, $\mathfrak{S}$, which, together with $\mathfrak{D}_1\mathfrak{D}_2$ and $\mathfrak{P}_i$, gives a set of 9 base-points of a cubic pencil.

The general curve $C^{(14)}$ has two points, $\mathfrak{D}_1$, $\mathfrak{D}_2$ of the multiplicity 7; 6 quadruple points $\mathfrak{P}_i$, and passes through $\mathfrak{S}$.

2). 3 double-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, and two simple fundamental points, $\mathfrak{P}_1$, $\mathfrak{P}_2$.

The correspondence is not fully determined thereby.

8th Order.—Centre-curve, $H^{(4)}$ of 4th order, with $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{P}_1$, $\mathfrak{P}_2$ as simple points.

3 singular lines, $|\mathfrak{D}_1\mathfrak{D}_2|$, $|\mathfrak{D}_1\mathfrak{D}_3|$, $|\mathfrak{D}_2\mathfrak{D}_3|$, each with one associated singular point, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$, and

1 singular conic through the 5 fundamental points, with one associated point $\mathfrak{S}^{(2)}$.

The general curve $C^{(8)}$ has 3 quadruple points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, 3 double-points, $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{S}^{(2)}$, and passes through the 3 singular points, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$.

3). 1 triple-point $\mathfrak{T}$ and 5 simple fundamental points $\mathfrak{P}_i$.

The correspondence is not fully determined thereby.

10th Order.—Centre-curve $H^{(4)}$ of 4th order, with $\mathfrak{T}$ as double-point and $\mathfrak{P}_i$ as simple points.

5 singular lines $s_i = |\mathfrak{T}\mathfrak{P}_i|$, each with one associated singular point $\mathfrak{S}_i$.

The general curve $C^{(10)}$ has a point of the multiplicity 7 in $\mathfrak{T}$, 5 triple points in $\mathfrak{P}_i$, and passes through $\mathfrak{S}_i$ as simple points.

§22.—*Nets of Quintics.*

Number of fixed points of intersection, $Q = n^2 - 2 = 23$.

Number of elements of determination, $P = \frac{(n-1)(n+4)}{2} = 18$.

The quintic net gives the following univalent correspondences:

1). 5 double-points, $\mathfrak{D}_i$, and 3 simple points, $\mathfrak{P}_i$; fully determined.

20th Order.—Centre-curve, $H^{(10)}$ of 10th order, with 5 triple-points, $\mathfrak{D}_i$, and 3 double-points, $\mathfrak{P}_i$.

One singular conic through the 5 double-points, $\mathfrak{D}_i$, with one associated singular point, $\mathfrak{S}^{(2)}$.

General curve, $C^{(20)}$, with 5 8-ple points, $\mathfrak{D}_i$, 3 quintuple points, $\mathfrak{P}_i$, and one double-point, $\mathfrak{S}^{(2)}$.

2). 1 triple-point, $\mathfrak{T}$, 2 double-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, and 6 simple points, $\mathfrak{P}_i$; fully determined.

22d Order.—Centre-curve, $H^{(10)}$ of 10th order, with $\mathfrak{T}$ as quintuple point, 2 triple-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, and 6 double-points, $\mathfrak{P}_i$.

2 singular lines, $|\mathfrak{T}\mathfrak{D}_1|$ and $|\mathfrak{T}\mathfrak{D}_2|$, each with one associated singular point, $\mathfrak{S}_1$, $\mathfrak{S}_2$.

General curve, $C^{(22)}$, with 13-ple point, $\mathfrak{T}$, 2 9-ple points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, 6 quintuple points, $\mathfrak{P}_i$, and 2 simple points, $\mathfrak{S}_1$, $\mathfrak{S}_2$.

3). 1 triple-point, $\mathfrak{T}$, 3 double-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, and 2 simple points, $\mathfrak{P}_1$, $\mathfrak{P}_2$; undetermined.

13th Order.—Centre-curve, $H^{(5)}$ of 5th order, with double-point, $\mathfrak{T}$, and simple points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{P}_1$, $\mathfrak{P}_2$.

3 singular lines, $|\mathfrak{T}\mathfrak{D}_1|$, $|\mathfrak{T}\mathfrak{D}_2|$, $|\mathfrak{T}\mathfrak{D}_3|$, each with one associated singular point, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$, and

2 singular conics, each passing through $\mathfrak{T}$, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$ and one of the simple points, $\mathfrak{P}_i$, each with 2 associated singular points, $\mathfrak{S}_1^{(2)}$, $\mathfrak{S}_2^{(2)}$; $\mathfrak{S}_{11}^{(2)}$, $\mathfrak{S}_{21}^{(2)}$.

General curve, $C^{(13)}$, with 8-ple point, $\mathfrak{T}$, 3 quintuple points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, 2 triple-points, $\mathfrak{P}_1$, $\mathfrak{P}_2$, 4 double-points, $\mathfrak{S}_1^{(2)}$, $\mathfrak{S}_2^{(2)}$, $\mathfrak{S}_{11}^{(2)}$, $\mathfrak{S}_{21}^{(2)}$, and 3 simple points, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$.

4). 1 quadruple point, $\mathfrak{Q}$, and 7 simple points, $\mathfrak{P}_i$; undetermined.

17th Order.—Centre-curve, $H^{(5)}$ of 5th order, with triple-point, $\mathfrak{Q}$, and 7 simple points, $\mathfrak{P}_i$.

7 singular lines, $|\mathfrak{Q}\mathfrak{P}_i|$, each with one associated point, $\mathfrak{S}_i$.

General curve, $C^{(17)}$, with 13-ple point, $\mathfrak{Q}$, 7 quadruple points, $\mathfrak{P}_i$, and 7 simple points, $\mathfrak{S}_i$.

§23.—*Nets of Sextics.*

$$Q = 34. \qquad P = 25.$$

1).	1 triple	point,	6	double points,	1	simple point;	fully determined.	35th	order.
2).	2 “	“	3	“	4	“	“ “	30th	“
3).	2 “	“	4	“	–	“	undetermined.	18th	“
4).	3 “	“	–	“	7	“	fully determined.	32d	“
5.)	3 “	“	1	“	3	“	undetermined.	20th	“
6).	1 quadruple	“	3	“	6	“	fully determined.	32d	“
7).	1 “	“	3	“	2	“	undetermined.	27th	“
8).	1 quintuple	“	–	“	9	“	“	26th	“
etc.							etc.		

B.—MULTIVALENT CORRESPONDENCES.

§24.—*Trivalent Cubic Correspondence*, $n = 2$.

A net of conics, which have no common point of intersection, determines a trivalent cubic correspondence, which has a cubic as its centre-curve, $H^{(3)}$.

To any point $\mathfrak{x}$ of the plane correspond 3 points $\mathfrak{y}_1$, $\mathfrak{y}_2$, $\mathfrak{y}_3$, which, when $\mathfrak{x}$

moves along a straight line g, produce a cubic curve $C^{(3)}$. This cubic $C^{(3)}$ intersects the straight line g in 3 self-corresponding points.

In general, a line g contains no pairs of conjugate or mutually corresponding points, But if a line g_0 contains one pair of points which correspond to one another, the line g_0 contains an infinite number of pairs of mutually corresponding points, which produce a *quadratic involution* on the line g_0.

Such a line may be called a *self-corresponding line.*

To any one of the self-corresponding lines g_0 corresponds (besides g_0) a conic; and all the self-corresponding lines g_0 envelop a curve of the third class.

For, the curve $C^{(3)}$, corresponding to a straight line g, is produced as the locus of points of intersection of two conic pencils of the net, which have one conic in common, and are put into correspondence by the points of the line g, so that to a conic of the one pencil always correspond two conics of the other pencil. These two conic pencils cut the line g in two quadratic involutions, which have as their common pair of elements the two points of intersection of g with the conic, which is common to both conic pencils.

If the line g contains a pair of mutually corresponding points, this is a common pair of elements of the two quadratic involutions also, and these involutions, therefore, are identically the same; that is, all their pairs of elements are mutually corresponding points on g, and g is a self-corresponding ray.

All the self-corresponding rays passing through a point $\mathfrak{x}$ contain a point corresponding to $\mathfrak{x}$ in their quadratic involution, and therefore in the cubic correspondence. Thus they are the three lines joining $\mathfrak{x}$ with its corresponding points $\mathfrak{y}_1$, $\mathfrak{y}_2$, $\mathfrak{y}_3$. Hence the self-corresponding rays envelop a curve of third class, q. e. d.

§25.—*Bivalent Correspondence of 4^{th} Order*, $n = 3$.

6 points, $\mathfrak{P}_1, \mathfrak{P}_2 \ldots \mathfrak{P}_6$ on a conic K determine as simple fundamental points of a cubic net a bivalent correspondence which has the conic K as singular curve, and therefore is of 4th order.

With the singular conic K is associated a singular point $\mathfrak{S}^{(2)}$.

This correspondence is *perspective*, with the singular point $\mathfrak{S}^{(2)}$ as the *centre of perspectivity.* The two points $\mathfrak{y}_1$, $\mathfrak{y}_2$, corresponding to $\mathfrak{x}$, lie on the straight line connecting $\mathfrak{x}$ with $\mathfrak{S}^{(2)}$. These lines of the pencil $\mathfrak{S}^{(2)}$ are *self-corresponding lines,* and contain *cubic involutions* of conjugate or mutually corresponding points.

The centre-curve $H^{(4)}$ is a quartic, passing through the 6 fundamental points.

The general curve $C^{(4)}$ passes through the 6 fundamental points, and has the singular point $\mathfrak{S}^{(2)}$ as a double-point.

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